International Journal of Theoretical Physics, Vol. 39, No. 4, 2000

Lie–Bäcklund Symmetries of Two-Dimensional *SU*(2) Yang–Mills System and Nonhomogeneous Lax Pair

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Received August 9, 1999

We obtain the Lie–Bäcklund-type symmetries of the two-dimensional *SU*(2) Yang–Mills equation with the help of a generalized formal series method. Both (x, t)-dependent and independent symmetries are obtained and it is shown that they form a closed algebra. Finally a nonhomogeneous Lax equation is derived using these symmetries.

1. INTRODUCTION

Integrable nonlinear systems have many interesting and special properties which are not shared by general class of partial differential equations.[(1)] One of them is the existence of an infinite number of conserved quantities and symmetries.[(2,3)] An important nonlinear equation which occurs in model quantum chromodynamics is the equation governing the *SU*(2) Yang–Mills theory in two dimension. The existence of a Lax pair for the system was proved by Ahmad and Roy Chowdhury[(4)] through the prolongation theory, which is an important step in proving the complete integrability of the system. Here we derive the general class of Lie–Bäcklund-type symmetries by adopting a generalized formal series approach advocated by Lou.[(5)]

In the last part we show how these symmetries can be effectively used to generate an alternative nonhomogeneous Lax equation for the system.

2. FORMULATION

The set at equations under consideration can be written as

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0020-7748/00/0400-1101$18.00/0

$$
\begin{aligned}
A^1_{xxt} + A^1_x - A^2_{xx}A^3 + A^2A^3_{xx} &= 0 \\
A^2_{xxt} + A^2_x - A^3_{xx}A^1 + A^3A^1_{xx} &= 0 \\
A^3_{xxt} + A^3_x - A^1_{xx}A^2 + A^1A^2_{xx} &= 0
\end{aligned} \tag{1}
$$

where A^i ($i = 1, 2, 3$) are the $SU(2)$ isocomponents of the gauge field. To set up a compact notation we rewrite the above set as

$$A^i_{xxt} + K^i(A^i, A^i_x, \ldots) = 0 \tag{2}$$

with $i = 1, 2, 3$ and

$$
\begin{aligned}
K^{(1)} &= A^1_x - A^2_{xx}A^3 + A^2A^3_{xx} \\
K^{(2)} &= A^2_x - A^3_{xx}A^1 + A^3A^1_{xx} \\
K^{(3)} &= A^3_x - A^1_{xx}A^2 + A^1A^2_{xx}
\end{aligned} \tag{3}
$$

The symmetry transformation for A^i can be written as

$$A^i \to A^i + \epsilon\sigma^i \tag{4}$$

so that the linearized equation is written as

$$\frac{\partial}{\partial\varepsilon}(A^i + \varepsilon\sigma^i)_{xxt}\Big|_{\varepsilon=0} + \frac{\partial}{\partial\varepsilon}K^i(A^i + \varepsilon\sigma^i, \ldots)\Big|_{\varepsilon=0} = 0$$

Written in full it reads

$$\begin{pmatrix}\sigma^1\\ \sigma^2\\ \sigma^3\end{pmatrix}_{xxt} + K'\begin{pmatrix}\sigma^1\\ \sigma^2\\ \sigma^3\end{pmatrix} = 0 \tag{5}$$

where

$$K' = \begin{pmatrix}
\dfrac{\delta K^1}{\delta A^1} & \dfrac{\delta K^1}{\delta A^2} & \dfrac{\delta K^1}{\delta A^3} \\
\dfrac{\delta K^2}{\delta A^1} & \dfrac{\delta K^2}{\delta A^2} & \dfrac{\delta K^2}{\delta A^3} \\
\dfrac{\delta K^3}{\delta A^1} & \dfrac{\delta K^3}{\delta A^2} & \dfrac{\delta K^3}{\delta A^3}
\end{pmatrix} \tag{6}$$

where

$$\frac{\delta K^i}{\delta A^j} = \sum_{l=0} \frac{\delta K^i}{\delta A^i_l}\,\partial_l \tag{7}$$

with $A^i_l = \partial^l A^i/\partial x^l$.

In the generalized series approach we set

$$\sigma = \begin{pmatrix} \sigma^1 \\ \sigma^2 \\ \sigma^3 \end{pmatrix} = \sum_{K=0}^{\infty} f^{(-k)}\sigma[K] \tag{8}$$

with $f^{(-1)}(t) = \partial_t^{-1} f(t)$, $f^{(-2)}(t) = \partial_t^{-2}$, etc., $f(t)$ being an arbitrary function of t. Substituting (8) in equation (5), we get

$$\sum_K \{ f^{(-k+1)}\sigma_{xxt}[K-1] + f^{(-k+2)}\sigma_{xx}[K-1] + f^{(-K+1)}K'\sigma[K-1]\} = 0 \tag{9}$$

Equating coefficients of $f^{(-K+1)}$, we get

$$\sigma^{ixt}[K-1] + \sigma^{ix}[K] + K'\sigma[K-1] = 0$$

or

$$\sigma[K] = (-\partial_t - \partial_x^{-2}K')\sigma[K-1] \tag{10}$$

yielding a recursion relation for $\sigma[K]$.

For $K = 0$ one gets

$$\begin{pmatrix} \sigma^1_{xx}[0] \\ \sigma^2_{xx}[0] \\ \sigma^3_{xx}[0] \end{pmatrix} = 0 \tag{11}$$

A simple implication of this equation is $\sigma[0] = x\,h(t)$ or $\sigma = H(t)$, where h and H are three-component vectors and a function of t only.

For $\sigma = H(t)$ we get

$$\sigma[1] = \begin{pmatrix} -\partial_t & -A^3 & A^2 \\ A^3 & -\partial_t & -A^1 \\ -A^2 & A^1 & -\partial_t \end{pmatrix} H(t) \tag{12}$$

Plugging this in the recursion relation (10), we get

$$\sigma[2] = \begin{pmatrix} \partial_t & 2A^3 & -2A^2 \\ -2A^3 & \partial_t & 2A^1 \\ 2A^2 & -2A^1 & \partial_t \end{pmatrix} \partial_t H(t) \tag{13}$$

Proceeding in the same fashion, we get

$$\sigma[K] = (-1)^K \begin{pmatrix} \partial_t & KA^3 & -KA^2 \\ -KA^3 & \partial_t & KA^1 \\ KA^2 & -KA^1 & \partial_t \end{pmatrix} \partial_t^{K-1} H(t) \tag{14}$$

$$= (-1)^K M_K \, \partial_t^{K-1} H(t)$$

so that

$$\sigma = \sum_K (-1)^K \, \partial_t^{(-K)} f(t) M_K \, \partial_t^{K-1} H(t) \tag{15}$$

Let us consider some special situations for different choices of $H(t)$.

For

$$H(t) = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \cdot t, \qquad (\alpha, \beta, \gamma) \text{ constant}$$

so that

$$\sigma_1 = \begin{pmatrix} f\alpha t - f^{(-1)} (\alpha + A^3\beta t - A^2\gamma t) + 2f^{(-2)} (A^3\beta - A^2\gamma) \\ f\beta t - f^{(-1)} (\beta + A^1\gamma t - A^3\alpha t) + 2f^{(-2)} (A^1\gamma - A^3\alpha) \\ f\gamma t - f^{(-1)} (\gamma + A^2 \alpha t - A^1\beta t) + 2f^{(-2)} (A^2\alpha - A^1\beta) \end{pmatrix} \tag{16}$$

it may be verified that σ_1 is a solution of the linearized equation (5). In the second case

$$H(t) = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \cdot t^2$$

which ultimately yields

$$\sigma_2 = \begin{pmatrix} f\alpha t^2 - f^{(-1)}(2\alpha t + A^3\beta t^2 - A^2\gamma t^2) \\ \quad + 2f^{(-2)}(\alpha + 2A^3\beta t - 2A^2\gamma t) - 2f^{(-3)}(3A^3\beta - 3A^2\gamma) \\ f\beta t^2 - f^{(-1)}(2\beta t + A^1\gamma t^2 - A^3\alpha t^2) \\ \quad + 2f^{(-2)}(\beta + 2A^1\gamma t - 2A^3\alpha t) - 2f^{(-3)}(3A^1\gamma - 3A^3\alpha) \\ f\gamma t^2 - f^{(-1)}(2\gamma t + A^2\alpha t^2 - A^1\beta t^2) \\ \quad + 2f^{(-2)}(\gamma + 2A^2\alpha t - 2A^1\beta t) - 2f^{(-3)}(3A^2\alpha - 3A^1\beta) \end{pmatrix} \tag{17}$$

On the other hand, the general form can be written as

$$\sigma[0] = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \cdot t^n \tag{18}$$

$$\sigma_n = \begin{pmatrix} L_n\alpha - K_n(A^3\beta - A^2\gamma) \\ L_n\beta - K_n(A^1\gamma - A^3\alpha) \\ L_n\gamma - K_n(A^2\alpha - A^1\beta) \end{pmatrix} \tag{19}$$

with

$$L_n(t) = \sum_{j=0}^{n} (-1)^j f^{(-j)}\, \partial_t^j(t^n)$$

$$K_n(t) = \sum_{j=0}^{n} (-1)^j (j+1) f^{-(j+1)} \cdot \partial_t^j(t^n) \tag{20}$$

On the other hand, due to translational symmetry of the equation in space and time, two symmetries are

$$\theta_1 = \begin{pmatrix} A_x^1 \\ A_x^2 \\ A_x^3 \end{pmatrix} = A_x, \qquad \theta_2 = \begin{pmatrix} A_t^1 \\ A_t^2 \\ A_t^3 \end{pmatrix} = A_t \tag{21}$$

We now compute the Jacobi bracket[6] of the two symmetries σ_i, θ_j, as

$$\{\sigma_i, \theta_j\} = \ni(\sigma_i)[\theta_j] - \ni(\theta_j)[\sigma_i] \tag{22}$$

where $\ni(\sigma_i)$ stands for

$$\ni(\sigma_i) = \begin{pmatrix} \dfrac{\delta\sigma_i^1}{\delta A_1} & \dfrac{\delta\sigma_i^1}{\delta A_2} & \dfrac{\delta\sigma_i^1}{\delta A_3} \\ \dfrac{\delta\sigma_i^2}{\delta A_1} & \dfrac{\delta\sigma_1^2}{\delta A_2} & \dfrac{\delta\sigma_i^2}{\delta A_3} \\ \dfrac{\delta\sigma_i^3}{\delta A_1} & \dfrac{\delta\sigma_i^3}{\delta A_2} & \dfrac{\delta\sigma_i^3}{\delta A_3} \end{pmatrix} \tag{23}$$

Then one can easily evaluate and obtain

$$\{A_t, \sigma_1\} = \begin{pmatrix} \alpha L_{1t} - (A^3\beta - A^2\gamma)L_1 \\ \beta L_{1t} - (A^1\gamma - A^3\alpha)L_1 \\ \gamma L_{1t} - (A^2\alpha - A^1\beta)L_1 \end{pmatrix} = \nu_{11} \quad \text{(say)} \tag{24}$$

It is now easy to demonstrate that ν_{11} is a solution of the linearized equation (5). We now define a set of such symmetries as

$$\nu_{n,(m-1)} = \begin{pmatrix} \alpha L_{n,mt} - (A^3\beta - A^2\gamma)L_{n,(m-1)t} \\ \beta L_{n,mt} - (A^1\gamma - A^3\alpha)L_{n,(m-1)t} \\ \gamma L_{n,mt} - (A^2\alpha - A^1\beta)L_{n,(m-1)t} \end{pmatrix} \tag{25}$$

where $L_{n,mt}$ denotes the mth derivative of L_n with respect to time, that is, $\partial^m L_n/\partial_t m$. It is then an immediate consequence that

$$\{A_t, \nu_{n,m}\} = \nu_{n,(m+1)} \tag{26}$$

So they are generated recursively. On the other hand, one can also deduce that

$$\{\sigma_m, \sigma_n\} = 0 \tag{27}$$

So they commute with respect to Jacobi bracket.

Lastly we may add a few comments for the situation when

$$\sigma[0] = xh(t) \qquad \text{with} \quad h(t) = \begin{pmatrix} p(t) \\ q(t) \\ r(t) \end{pmatrix}$$

From equation (10) we can at once evaluate

$$\begin{aligned} \sigma^1[1] &= (-xp_t - xA^3q + xA^2r) + 2\left(-\frac{x^2}{4}\,p + q\partial_x^{-1}\,A^3 - r\partial_x^{-1}A^2\right) \\ \sigma^2[1] &= (-xq_t + xA^3p + xA^1r) + 2\left(-q\,\frac{x^2}{4} + r\partial_x^{-1}A^1 - p\partial_x^{-1}A^3\right) \\ \sigma^3[1] &= (-xr_t - xA^1q - xA^2p) + 2\left(-r\,\frac{x^2}{4} + p\partial_x^{-1}A^2 - q\partial_x^{-1}A^1\right) \end{aligned} \tag{29}$$

The expressions for $\sigma^i[2]$ and other $\sigma^i[K]$ become highly complicated and are not reproduced here, but the series in equation (8) is not truncated.

3. NONHOMOGENEOUS LAX PAIR

Let us now refer to ref. 4, where a Lax pair was obtained for equation (1) via a prolongation technique. Such a pair can be written as

$$Y_n = Fy; \qquad Y_t = Gy \tag{30}$$

Y is a three-component vector and

$$F = \begin{pmatrix} 0 & -A^3_{xx} & A^2_{xx} \\ A^3_{xx} & 0 & -A^1_{xx} \\ -A^2_{xx} & A^1_{xx} & 0 \end{pmatrix}; \qquad G = \begin{pmatrix} 0 & A^3 & -A^2 \\ -A^3 & 0 & A^1 \\ A^2 & -A^1 & 0 \end{pmatrix}$$

This set can also be written as

$$L_1 Y = 0, \qquad L_2 Y = 0 \qquad \text{with} \qquad L_1 = \partial_x - F, \qquad L_2 = \partial_t - G \tag{31}$$

and the nonlinear system is equivalent to $[L_1, L_2] = 0$. In the following our motivation is to search for a Lax pair of the form

$$L_1 Y = fY, \qquad L_2 Y = gY \tag{32}$$

Thus, that the consistency $L_1 L_2 Y = L_2 L_1 Y$ again leads to the nonlinear equation, which implies that

$$(L_1 g - L_2 f)Y = 0 \tag{33}$$

Let us now choose $f = \sigma'_{ixx}$, where σ'_i stands for a Lie–Bäcklund symmetry of equation (1) satisfying

$$\begin{aligned} &\sigma^1_{ixxt} + \sigma^1_{ix} - A^2_{xx}\sigma^3_i - A^3\sigma^2_{ixx} + A^3_{xx}\sigma^2_i + A^2\sigma^3_{ixy} = 0 \\ &\sigma^2_{ixxt} + \sigma^2_{ix} - A^3_{xx}\sigma^1_i - A^1\sigma^3_{ixx} + A^1_{xx}\sigma^3_i + A^3\sigma^1_{ixx} = 0 \\ &\sigma^3_{ixxt} + \sigma^3_{ix} - A^1_{xx}\sigma^2_i - A^2\sigma^1_{ixx} + A^2_{xx}\sigma^1_i + A^1\sigma^2_{ixx} = 0 \end{aligned} \tag{34}$$

Then equation (33) can be written as

$$g_x y + gFy - Fgy - \sigma_{iixxt} y - \sigma_{ixx} GY + G\sigma_{ixx} y = 0 \tag{35}$$

Let us assume g be of general from (g_{ij}),

$$g = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} \tag{36}$$

whence from equation (35) we obtain

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix}_x = \begin{pmatrix} 0 & A^1_{xx} & A^2_{xx} \\ -A^1_{xx} & 0 & A^3_{xx} \\ -A^2_{xx} & -A^3_{xx} & 0 \end{pmatrix} \cdot \begin{pmatrix} A \\ B \\ C \end{pmatrix} \tag{37}$$

where $A = g_{12} - g_{21}$, $B = g_{13} - g_{31}$, $C = g_{23} - g_{32}$. If we choose the trivial solution $A = B = C = 0$, then we get

$$R_x + MR = T \tag{38}$$

where $R = (g_{11}, g_{22}, g_{33}, g_{12}, g_{13}, \mathrm{g}_{23})^t$, $T = \{T_i\}$, $i = 1, \ldots, 6$, and M is a 6×6 matrix. Their explicit forms are as follows,

$$
\begin{aligned}
T_1 &= -\sigma^1_{ix} + A^2_{xx}\sigma^3_i + A^3\sigma^2_{ixx} - A^3_{xx}\sigma^2_i - A^2\sigma^3_{ixx} \\
T_2 &= -\sigma^2_{ix} + A^3_{xx}\sigma^1_i + A^1\sigma^3_{ixx} - A^1_{xx}\sigma^3_i - A^3\sigma^1_{ixx} \\
T_3 &= -\sigma^3_{ix} + A^1_{xx}\sigma^2_i + A^2\sigma^1_{ixx} - A^2_{xx}\sigma^1_i - A^1\sigma^2_{ixx} \qquad (39) \\
T_4 &= (\sigma^1_{ixx} - \sigma^2_{ixx})\, A^3 \\
T_5 &= (\sigma^3_{ixx} - \sigma^1_{ixx})\, A^2 \\
T_6 &= (\sigma^2_{ixx} - \sigma^3_{ixx})\, A^1
\end{aligned}
$$

and

$$
M = \begin{pmatrix}
0 & 0 & 0 & 2A^3_{xx} & -2A^2_{xx} & 0 \\
0 & 0 & 0 & -2A^3_{xx} & 0 & 2A^1_{xx} \\
0 & 0 & 0 & 0 & 2A^2_{xx} & -2A^1_{xx} \\
-A^3_{xx} & A^3_{xx} & 0 & 0 & A^1_{xx} & -A^2_{xx} \\
A^2_{xx} & 0 & -A^2_{xx} & -A^1_{xx} & 0 & A^3_{xx} \\
0 & -A^1_{xx} & A^1_{xx} & A^2_{xx} & -A^3_{xx} & 0
\end{pmatrix} \qquad (40)
$$

Equation (38) is a linear equation and can be explicitly solved. So we have shown that one can obtain (f, g) explicitly in terms of the field variables and hence a nonhomogeneous Lax pair can be realized.

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